TOPICS IN STATISTICAL PHYSICS AND PROBABILITY THEORY HOMEWORK SHEET 2

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To hand in by July 10 to the instructor's mailbox at Schreiber building.

In problems (i)-(iii) we consider the setup of infinite-volume Gibbs measures on configurations $\Omega := \{\varphi \,|\, \varphi : \mathbb{Z}^d \to S\}$ given in the accompanying note.

(i) Let $\mathbb{P}$ be an *extremal* Gibbs measure on Ω. Let (Λ_n) be an increasing sequence of finite sets in $\mathbb{Z}^d$ which increases to $\mathbb{Z}^d$ (i.e., $\Lambda_n \subseteq \Lambda_{n+1}$ and $\cup\Lambda_n = \mathbb{Z}^d$).
Prove that the measures $\mathbb{P}^{\varphi}_{\Lambda_n}$ converge to $\mathbb{P}$, $\mathbb{P}$-almost surely (that is, one samples φ from $\mathbb{P}$ and then considers $\mathbb{P}^{\varphi}_{\Lambda_n}$ for all n with this fixed φ, and convergence occurs for $\mathbb{P}$-almost every φ).
Remark: There exist *non-extremal* Gibbs measures $\mathbb{P}$ for which there is no $\eta : \mathbb{Z}^d \to S$ with $\mathbb{P}^{\eta}_{\Lambda_n}$ converging to $\mathbb{P}$. See Example 6.64 in the book of Friedli and Velenik for such an example for the 3-dimensional Ising model. There are open questions and conjectures regarding which Gibbs measures are given by such limits (see, e.g., the paper of Coquille https://arxiv.org/abs/1411.3265).

(ii) Prove that a translation-invariant Gibbs measure $\mathbb{P}$ is extremal within the set of translation-invariant Gibbs measures if and only if $\mathbb{P}$ is ergodic.
Remark: The exercise implies, in particular, that if a translation-invariant Gibbs measure is extremal (within all Gibbs measures) then it is ergodic. However, there are models in which not all ergodic Gibbs measures are extremal.
Hint: Birkhoff's ergodic theorem for $\mathbb{Z}^d$ shifts states the following:
Let $f : \Omega \to \mathbb{R}$ be integrable. Let $\mathcal{I}$ be the sigma-algebra of all translation-invariant events $A \subseteq \Omega$. Then for every translation-invariant probability measure $\mathbb{P}$ on Ω,

$$\lim_{L\to\infty} \frac{1}{(2L+1)^d} \sum_{v\in\{-L,\ldots,L\}^d} f(\theta_v\varphi) = \mathbb{P}(f \,|\, \mathcal{I}) \quad \mathbb{P}\text{-almost surely and in } L^1.$$

where $\theta_v\varphi$ is the configuration satisfying $(\theta_v\varphi)_w = \varphi(w-v)$.

(iii) Let $\mathbb{P}$ be a Gibbs measure. Prove that $\mathbb{P}$ is extremal if and only if for every $A \subseteq \Omega$ measurable,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{B\in\mathcal{F}_{\Lambda^c}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| = 0,$$

where we write $B \in \mathcal{F}_{\Lambda^c}$ to indicate that $B \subseteq \Omega$ is measurable and $1_B(\varphi) = 1_B(\varphi')$ whenever $\varphi_v = \varphi'_v$ for all $v \in \mathbb{Z}^d \setminus \Lambda$, and where we write $\lim_{\Lambda\uparrow\mathbb{Z}^d}$ to indicate a limit over all sequences (Λ_n) of finite sets which increase to $\mathbb{Z}^d$.
Remark: Thus extremal measures satisfy a form of *weak mixing*: 'far away' events are almost uncorrelated.

(iv) Consider the Ising model with free boundary conditions. That is, fixing $h, \beta \geqslant 0$, the model for each finite $\Lambda \subseteq \mathbb{Z}^d$ is the measure $\mathbb{P}^{\emptyset}_{\Lambda}$ on functions $\varphi : \Lambda \to \{-1,1\}$ given by

$$\mathbb{P}^{\emptyset}_{\Lambda}(\varphi) = \frac{1}{Z^{\emptyset}_{\Lambda}} \exp\left(\beta \sum_{u,v\in\Lambda,\, u\sim v} \varphi_u\varphi_v + h\sum_{v\in\Lambda}\varphi_v\right), \tag{1}$$

where, as usual, $Z^{\emptyset}_{\Lambda}$ is a normalization factor.
Let (Λ_n) be a sequence of finite sets in $\mathbb{Z}^d$ which increases to $\mathbb{Z}^d$. Prove that $\mathbb{P}^{\emptyset}_{\Lambda_n}$ converge to a limiting Gibbs measure $\mathbb{P}^{\emptyset}$ on functions $\varphi : \mathbb{Z}^d \to \{-1,1\}$ and that $\mathbb{P}^{\emptyset}$ is translation invariant.

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Clarification: We may view each measure $\mathbb{P}^{\emptyset}_{\Lambda_n}$ as a measure on functions $\varphi : \mathbb{Z}^d \to \{-1, 1\}$ which is supported on functions with some fixed value outside Λ_n, e.g., having $\varphi_v = 1$ when $v \notin \Lambda_n$. In this way all the $\mathbb{P}^{\emptyset}_{\Lambda_n}$ are measures on the same space and convergence in distribution is well defined.
Hint: This is Exercise 3.16 in the Friedli-Velenik book. Use the GKS inequality to first prove that for any finite $A \subseteq \mathbb{Z}^d$, $\mathbb{P}^{\emptyset}_{\Lambda_n}(\prod_{v\in A}\varphi_v)$ increases with n (Exercise 3.12).

(v) Consider the Ising model with free boundary conditions at zero magnetic field, that is, the model (1) with $h = 0$. To emphasize the dependence on temperature we now denote the finite volume Gibbs measures by $\mathbb{P}^{\emptyset}_{\Lambda,\beta}$ and the infinite-volume limit (of exercise (iv)) by $\mathbb{P}^{\emptyset}_{\beta}$. Define the critical inverse temperature β_c by

$$\beta_c := \inf(\beta \mid \inf_{v\in\mathbb{Z}^d} \mathbb{P}^{\emptyset}_{\beta}(\varphi_v\varphi_{\mathbf{0}}) > 0)$$

(it can be shown that this definition coincides with the definitions discussed in class). The following version of Simon's inequality is due to Lieb: Let $\Lambda \subseteq \mathbb{Z}^d$ be a finite connected set containing the origin $\mathbf{0}$ and let $v \in \mathbb{Z}^d \setminus \Lambda$. Then

$$\mathbb{P}^{\emptyset}_{\beta}(\varphi_v\varphi_{\mathbf{0}}) \leqslant \sum_{u\in\partial_{\text{int}}\Lambda} \mathbb{P}^{\emptyset}_{\Lambda,\beta}(\varphi_u\varphi_{\mathbf{0}})\mathbb{P}^{\emptyset}_{\beta}(\varphi_v\varphi_u),$$

where we write $\partial_{\text{int}}\Lambda := \{u \in \Lambda \mid \exists w \notin \Lambda, u \sim w\}$ for the internal vertex boundary of Λ.
Deduce that for every finite connected set Λ containing the origin one has

$$\sum_{u\in\partial_{\text{int}}\Lambda} \mathbb{P}^{\emptyset}_{\Lambda,\beta_c}(\varphi_u\varphi_{\mathbf{0}}) \geqslant 1. \tag{2}$$

Remark: This means that at the critical point the correlations cannot decay faster than polynomially. We have also seen that for $\beta < \beta_c$ the correlations decay exponentially whereas for $\beta > \beta_c$ they do not decay (by definition).
Hint: Show that if there exists some Λ for which (2) is violated then there is exponential decay of correlations. Apply this also at $\beta = \beta_c + \varepsilon$.

(vi) Let $U : \mathbb{R} \to \mathbb{R}$ be a C^2 function satisfying $U(x) = U(-x)$ and $\sup_x U''(x) < \infty$. The two-dimensional *random surface model with potential* U is specified as follows: For each finite $\Lambda \subseteq \mathbb{Z}^2$ and $\eta : \mathbb{Z}^2 \to \mathbb{R}$ the probability measure $\mathbb{P}^{\eta}_{\Lambda}$ on configurations $\varphi : \mathbb{Z}^2 \to \mathbb{R}$ is given by

$$d\mathbb{P}^{\eta}_{\Lambda}(\varphi) = \frac{1}{Z^{\eta}_{\Lambda}} \exp\left(-\sum_{\substack{u\sim v\\ \{u,v\}\cap\Lambda\neq\emptyset}} U(\varphi_u - \varphi_v)\right) \prod_{v\in\Lambda} d\varphi_v \prod_{v\in\mathbb{Z}^2\setminus\Lambda} d\delta_{\eta_v}(\varphi_v) \tag{3}$$

where δ_s is the Dirac delta measure at s, so that the measure $\mathbb{P}^{\eta}_{\Lambda}$ is supported on configurations φ which equal η outside Λ and where Z^{η}_{Λ} is chosen to normalize the measure to be a probability measure (and we assume that U satisfies sufficient integrability conditions to ensure that such normalization is possible).

(a) Mimic the proof of the Mermin-Wagner theorem given in class for the XY model to show the following: There exist positive $C = C(U)$ and $\alpha = \alpha(U)$ such that for all integer $L > 0$, all $0 < t < 1$ and all $\eta : \mathbb{Z}^2 \to \mathbb{R}$,

$$|\mathbb{P}^{\eta}_{\Lambda_L}(\exp(it\varphi_{(0,0)}))| \leqslant \frac{C}{L^{\alpha t^2}},$$

where $\Lambda_L := \{-L, -L+1, \dots, L\}^2$.
Remark: A similar argument shows that $\mathrm{Var}^{\eta}_{\Lambda_L}(\varphi_{(0,0)}) \geqslant c(U)\log(L)$ for some positive $c(U)$.

(b) Deduce that the model does not have any Gibbs measures in two dimensions.
Remark: The same holds in dimension $d = 1$. In dimensions $d \geqslant 3$ the model does admit Gibbs measures.